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Large-amplitude ion acoustic solitons in a warm magnetoplasma

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Abstract. We have studied the nonlinear propagation of ion acoustic waves in a warm magnetoplasma. The limits of different parameters for the existence of localised soliton solutions are obtained. An analytical expression is obtained for the small-amplitude limit of this wave.

1. Introduction

In the last two decades, interest has grown in studying the nonlinear solitary ion acoustic waves in plasmas using the reductive perturbation theory (Washimi and Taniuti 1966, Davidson 1972, Tappert 1972, Tagare 1972). Following the reductive perturbation method all the nonlinear terms of the equations of motion cannot be taken into account. Sagdeev (1966) demonstrated the nonlinear propagation of ion acoustic solitons in a cold plasma without using the reductive perturbation technique. Taking all the nonlinear terms into account, he showed that finite amplitude localised density humps with speed V, for $V_S < V < 1.6V_S$, where $V_S = (T_e/M_i)^{1/2}$ is the ion acoustic speed, can occur. The basic equations governing the dynamics of nonlinear solitary waves can be reduced to an equation in the form of the energy integral of a classical particle in a potential well. Analysing the potential, one can find the existence of localised solitons. Recently, Zakharov and Kuznetsov (1974) have studied the nonlinear finite but slow ion acoustic solitary waves in the presence of a magnetic field. Later, Shukla and Yu (1978) investigated the same problem with ion acoustic waves propagating obliquely to the magnetic field. Their studies have, however, been restricted to the small-amplitude limit in the sense that all the nonlinear terms are not taken into account. Recently Yu *et al* (1980) demonstrated the fully nonlinear ion acoustic solitons in a magnetoplasma with hot electrons and cold ions.

In this paper, we study the effects of ion temperature on the propagation of nonlinear ion acoustic solitons taking all the nonlinear terms into account. In § 2, we have reduced the basic equations into an equation analogous to the energy integral of a classical particle in a potential well. Section 3 contains the analysis of the Sagdeev potential to determine the necessary conditions for the existence of localised solitons. In § 4 we have briefly discussed the small-amplitude limit of the wave. Finally, our results are discussed in § 5.

2. Basic equations

We consider the propagation of a nonlinear ion acoustic wave in a plasma consisting of hot electrons and warm ions in the presence of a constant magnetic field $B_0\boldsymbol{z}$. The wave dynamics is governed by the following equations,

$$\partial_t n + \nabla\cdot(n\boldsymbol{v}) = 0 \tag{1}$$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{e}{M_i}\nabla\phi - \frac{1}{M_i n}\nabla p + (\boldsymbol{v}\times\boldsymbol{\Omega}_i) \tag{2}$$

$$n = n_e = n_0 \exp(e\phi/T_e) \tag{3}$$

where ϕ is the electrostatic potential of the wave, n, v, Ω_i ($=eB_0/M_iC$), p and M_i are the ion density, ion velocity, ion gyrofrequency, ion pressure and ion mass respectively. n_e and T_e are the electron density and temperature respectively. In equation (3), we have made use of the charge neutrality condition. In equation (2), we shall express the pressure term in terms of density by the use of the thermodynamic equation of state (Chen 1974),

$$p = C(nM_i)^\gamma \tag{4}$$

where C is a constant and γ is the ratio of the specific heats C_p/C_v. In this problem, we assume the ion motion to be three dimensional and neglect the variation of all quantities in the y direction. Equations (1)–(3) can now be written as follows:

$$\partial_t n + \partial_x(nv_x) + \partial_z(nv_z) = 0 \tag{5}$$

$$\partial_t v_x + (v_x\partial_x + v_z\partial_z)v_x = -(e/M_i)\partial_x\phi + v_y\Omega_i - \frac{T}{M_i(n_0^2 n)^{1/3}}\partial_x n \tag{6}$$

$$\partial_t v_y + (v_x\partial_x + v_z\partial_z)v_y = -v_x\Omega_i \tag{7}$$

$$\partial_t v_z + (v_x\partial_x + v_z\partial_z)v_z = -(e/M_i)\partial_z\phi - \frac{T}{M_i(n_0^2 n)^{1/3}}\partial_z n \tag{8}$$

$$n = n_0 \exp(e\phi/T_e) \tag{9}$$

where the ion temperature is defined as $T = 5p_0/3n_0$, p_0 and n_0 are, respectively, the equilibrium ion pressure and density. In this problem, the number of degrees of freedom for ions is three. Hence, for adiabatic compression, γ is taken to be $\frac{5}{3}$ (Chen 1974) in equations (6) and (8). The linear dispersion relation is deduced from equations (5)–(9) and is given by

$$\omega = k_z C_s(1 + k_x^2\rho_s^2)^{-1/2} \tag{10}$$

where $C_s = (K(T_e + T)/M_i)^{1/2}$ is the ion acoustic speed, K is the Boltzmann constant, k_x and k_z are wave vectors in the x and z directions respectively and $\rho_s = C_s/\Omega_i$ is the ion gyroradius. In the deduction of equation (10), we have assumed $\omega \ll \Omega_i$. The dispersion of the ion acoustic waves in this system is purely due to gyroradius effects. This dispersion can balance the nonlinear steepening of large-amplitude ion acoustic waves to evolve into solitary waves.

In order to study the nonlinear propagation of the ion acoustic waves, we assume that the wave is stationary in the moving frame defined by

$$\eta = l_x x + l_z z - Mt, \qquad \tau = t \tag{11}$$

where M is the velocity of the nonlinear wave and l_x and l_z are direction cosines along the x and z directions respectively. Thus $l_x^2 + l_z^2 = 1$.

Substituting equation (11) in equations (5)–(9) and then imposing the boundary conditions $n = 1$, $\phi = 0$, $v = 0$ at $\eta \to \pm\infty$, we obtain the following equation,

$$(\mathrm{d}/\mathrm{d}\eta)^2[\ln n + \tfrac{3}{2}\sigma n^{2/3} + M^2/2n^2] = \{-(l_z^2/M^2)[(n-1)n + \tfrac{3}{5}\sigma(n^{5/3}-1)n] + (n-1)\} \quad (12)$$

where we have normalised n to n_0, M to V_s, V_i $(i = x, y, z)$ to V_s, η to ρ $(= V_s/\Omega_i)$ and ϕ to T_e/e. The velocities V_x and V_z are defined in the laboratory frame and $\sigma = T/T_e$.

Multiplying both sides of equation (12) by

$$(\mathrm{d}/\mathrm{d}\eta)(\ln n + \tfrac{3}{2}\sigma n^{2/3} + M^2/2n^2)$$

and integrating, we obtain the following equation,

$$(\mathrm{d}n/\mathrm{d}\eta)^2 + \psi(n, M, l_z) = 0 \quad (13)$$

where

$$\begin{aligned}\psi(n) = [n^4/M^2(n^2 + \sigma n^{8/3} - M^2)^2]\{&l_z^2 n^2(n-1)^2 + 2M^2 n \\ &\times[(1-l_z^2)n\ln n - (n - l_z^2)(n-1)] + M^4(1-n)^2 \\ &+ \tfrac{6}{5}\sigma l_z^2 n^2(n^{8/3} - n^{5/3} - n + 1) - \tfrac{3}{5}\sigma l_z^2 M^2 n(3n^{5/3} - 5n + 2) \\ &+ \tfrac{9}{25}\sigma^2 l_z^2 n^2(n^{10/3} - 2n^{5/3} + 1) - \tfrac{3}{5}\sigma M^2 n^2(2n^{5/3} - 5n^{2/3} + 3)\}. \end{aligned} \quad (14)$$

Here, we have imposed the boundary condition $\mathrm{d}n/\mathrm{d}\eta = 0$ at $n = 1$. Equation (13) is in the form of the energy integral of a classical particle in a potential well. $\psi(n)$ in equation (14) is known as the Sagdeev or classical potential. We shall analyse the Sagdeev potential in the next section to determine the necessary conditions for the existence of stationary solitons. If we put $\sigma = 0$, then equation (14) is reduced to equation (10) of Yu *et al* (1980). Now for the soliton solution, $\psi(N) = 0$, where N is the amplitude of the soliton. Putting this condition in equation (14), we get the nonlinear dispersion relation given by

$$\begin{aligned} l_z^2 N^2(N-1)^2 &+ 2M^2 N[(1-l_z^2)N\ln N - (N-l_z^2)(N-1)] \\ &+ M^4(1-N)^2 + \tfrac{6}{5}\sigma l_z^2 N^2(N^{8/3} - N^{5/3} - N + 1) \\ &- \tfrac{3}{5}\sigma l_z^2 M^2 N(3N^{5/3} - 5N + 2) \\ &+ \tfrac{9}{25}\sigma^2 l_z^2 N^2(N^{10/3} - 2N^{5/3} + 1) - \tfrac{3}{5}\sigma M^2 N^2(2N^{5/3} - 5N^{2/3} + 3) = 0. \end{aligned} \quad (15)$$

This is a relation between the soliton amplitude and its speed.

3. Analysis

Now we analyse the Sagdeev potential $\psi(n)$ to determine the necessary conditions under which solitary wave solutions can exist. It is found from the analogy of the motion of a classical particle in a potential well that localised soliton solutions would be possible if $\psi(n)$ is negative between the points $n = 1$ and $n = N$. In the potential well, a particle entering from the left will move to the right side of the well $(\eta > 0)$ and is reflected at $n = N$, which then returns to $\eta = 0$, where $n = 1$, making a single transit. Therefore, the necessary conditions for the existence of localised wave solutions are $\psi(N)=\psi(1)=\psi_n(1)=0$. Now, in what follows, we study the behaviour of $\psi(n)$ at $n\simeq 1$ and $n\simeq N$. The Taylor expansion of the Sagdeev potential $\psi(n)$ at $n\simeq 1$ and $n\simeq N$ can be written respectively as

$$\begin{aligned}\psi(n\simeq 1)=&[1/M^2(1+\sigma-M^2)^2][(l_z^2-M^2)(1-M^2)\\&+2\sigma l_z^2-\sigma l_z^2M^2+\sigma^2 l_z^2-\sigma M^2](n-1)^2\\&+[2/M^2(1+\sigma-M^2)^3](1+\tfrac{2}{3}\sigma-2M^2)[(l_z^2-M^2)(1-M^2)+2\sigma l_z^2\\&-\sigma l_z^2M^2+\sigma^2 l_z^2-\sigma M^2](n-1)^2+[2/3M^2(1+\sigma-M^2)^2]\\&\times[3l_z^2-2M^2-M^2l_z^2+7\sigma l_z^2-\tfrac{4}{3}\sigma l_z^2M^2+4\sigma^2 l_z^2-\tfrac{8}{3}\sigma M^2](n-1)^3\end{aligned}\tag{16}$$

and

$$\begin{aligned}\psi(n\simeq N)=&[N^4/M^2(N^2+\sigma N^{8/3}-M^2)^2]\\&\times\{2l_z^2N(2N^2-3N+1)+2M^2[N(1-l_z^2)(2l_nN+1)\\&-(2N-l_z^2)(N-1)-N(N-l_z^2)]+2M^4(N-1)\\&+\tfrac{6}{15}\sigma l_z^2N(14N^{8/3}-11N^{5/3}-9N+6)\\&-\tfrac{6}{5}\sigma l_z^2M^2(4N^{5/3}-5N+1)+(18\sigma^2 l_z^2/75)N(8N^{10/3}-11N^{5/3}+3)\\&-\tfrac{2}{5}\sigma M^2N(11N^{5/3}-20N^{2/3}+9)\}(n-N).\end{aligned}\tag{17}$$

From equations (16) and (17), we cannot easily find out the necessary conditions for the existence of a solitary wave. Therefore, we have solved equation (15) numerically for M for different values of N, I_z and σ. $\psi(n)$ is then calculated from equation (14) for each set of values obtained from (15). We have found that for some sets of values of M, N, I_z and σ, $\psi(n)$ gives the soliton solution. The width Δ of this soliton can be calculated from the maximum depth d of the Sagdeev potential $\psi(n)$ and is given by (Bujarbarua and Schamel 1981)

$$\Delta=N/\sqrt{d}.\tag{18}$$

4. Small-amplitude limit

We now discuss the small-amplitude case of the soliton. For $N\simeq 1$, equation (13) can be written in the following form,

$$\tfrac{1}{2}(\mathrm{d}\delta n/\mathrm{d}\eta)^2+X_1\delta n^2+X_2\delta n^3=0\tag{19}$$

where

$$\begin{aligned}X_1=&(1/M^6)\{[l_z^2-M^2(1+l_z^2)+M^4+2\sigma l_z^2-\sigma l_z^2M^2+\sigma^2 l_z^2-\sigma M^2]\\&\times[1+2(1+\sigma)/M^2+3(1+\sigma)^2/M^4+4(1+\sigma)^3/M^6]\}\end{aligned}\tag{20}$$

and

$$\begin{aligned}X_2=&(1/M^6)\{[l_z^2-M^2(1+l_z^2)+M^4+2\sigma l_z^2-\sigma l_z^2M^2+\sigma^2 l_z^2-\sigma M^2]\\&\times[(4/M^2)(1+\tfrac{4}{3}\sigma)+12(1+\sigma)(1+\tfrac{4}{3}\sigma)/M^4+24(1+\sigma)^2/M^6\\&\times(1+\tfrac{4}{3}\sigma)+4+8(1+\sigma)/M^2+12(1+\sigma)^2/M^4+16(1+\sigma)^3/M^6]\end{aligned}$$

$$+[1+2(1+\sigma)/M^2+3(1+\sigma)^2/M^4+4(1+\sigma)^3/M^6]$$
$$\times[2l_z^2-\tfrac{2}{3}M^2(2+l_z^2)+\tfrac{14}{3}\sigma l_z^2-\tfrac{8}{9}\sigma l_z^2M^2+\tfrac{8}{3}\sigma^2 l_z^2-\tfrac{16}{9}\sigma M^2]\} \quad (21)$$

and $\delta n = n-1 = O(\varepsilon)$.

Equation (19) can be integrated once to yield the well known soliton solution (Washimi and Taniuti 1966, Yu *et al* 1980),

$$\delta n = \delta N \operatorname{sech}^2[(-X_1/2)^{1/2}\eta], \qquad X_1<0 \quad (22)$$

where $\delta N = -(X_1/X_2)$, and we have imposed the boundary conditions $d\delta n/d\eta = \delta n = 0$ at $\eta \to \pm\infty$.

5. Discussion

Our numerical results are plotted in figures 1–5. As it is well known that for ion temperature comparable to electron temperature, ion acoustic waves are heavily Landau damped (Chen 1974), we have chosen the values of σ less than 0.2867. Also numerical results show that ion acoustic solitons exist in a warm magnetoplasma for $0.7 \leqslant l_z \leqslant 0.8$. This indicates that the direction of the wave vector must be restricted to this small range so as to obtain solitary waves.

From figure 1, it is clear that the velocity of the soliton increases with the increase of σ for fixed values of l_z.

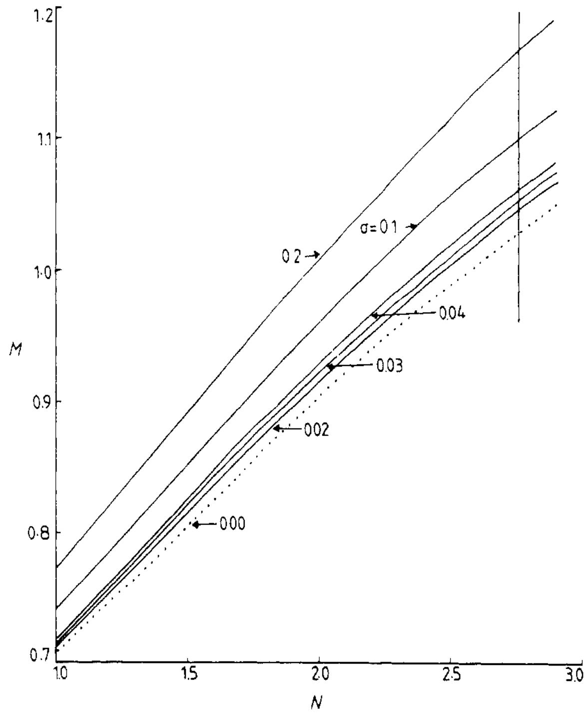

Figure 1. The Mach number of the soliton as a function of its amplitude for different values of σ at fixed $l_z = 0.7$. The soliton solution exists only to the left of the vertical line. The dotted curve corresponds to a cold plasma.

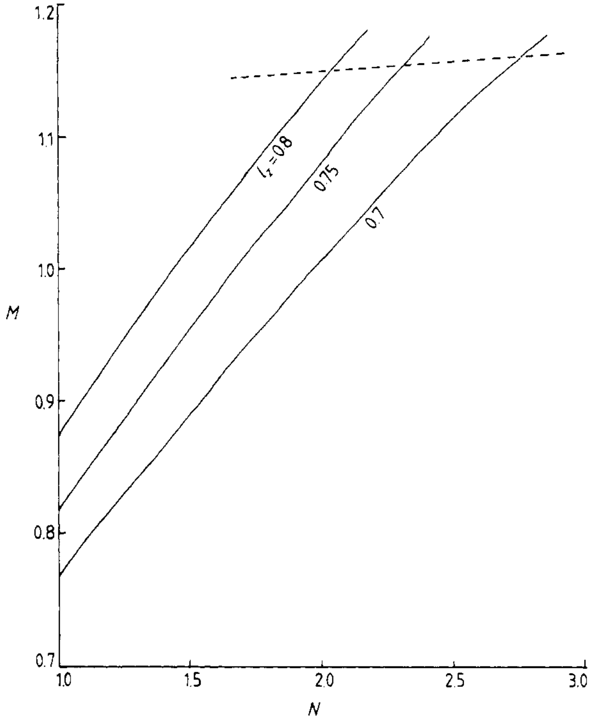

Figure 2. The Mach number of the soliton as a function of its amplitude for different values of l_z at fixed $\sigma = 0.2$. The localised soliton solution exists only below the broken line.

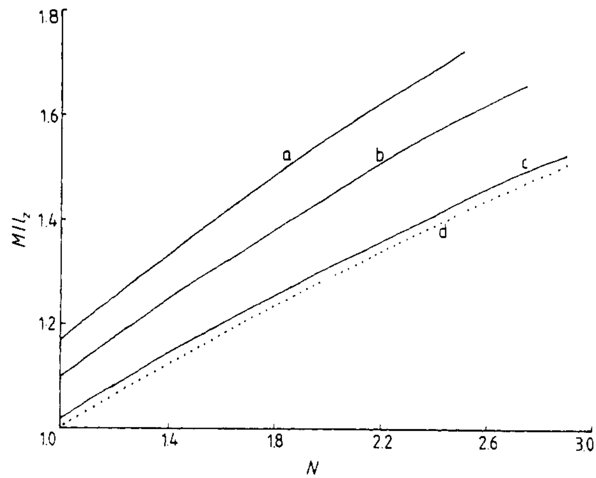

Figure 3. The variation of M/l_z with the amplitude of the soliton for different values of σ: a, $\sigma = 0.2$, $l_z = 0.75$; b, $\sigma = 0.2$, $l_z = 0.7$; c, $\sigma = 0.02$, $l_z = 0.7$, and d, $\sigma = 0$, $l_z = 0.7$.

The velocity of the soliton increases with l_z for fixed values of σ as is evident from figure 2. On the other hand, the maximum amplitude decreases with the increase of l_z for fixed ion temperatures and velocities of the soliton.

In figure 3, we have plotted M/l_z against N. The ratio of the Mach number to l_z is always found to be greater than unity.

In figures 4 and 5, we have plotted the width of the soliton as a function of its amplitude for fixed values of l_z and σ. In figure 4 the broken curve corresponds to a cold plasma. The finite temperature of the ions increases the width of the soliton for fixed amplitude and l_z. It is seen from figure 5 that the width of the soliton decreases with the increase of l_z for fixed amplitude and σ.

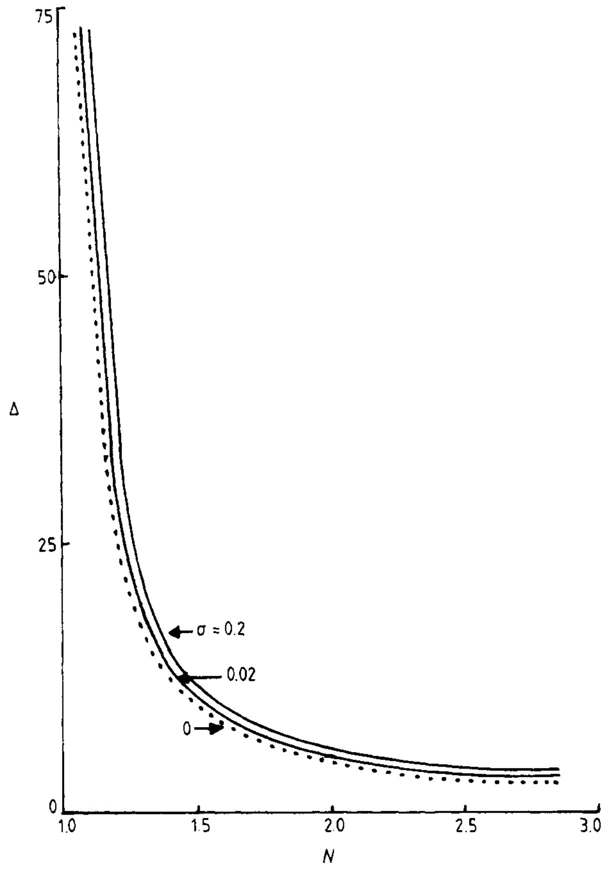

Figure 4. The variation of the width of the soliton with its amplitude at fixed $l_z = 0.7$ for different values of σ.

The numerical calculations of equation (20) show that X_1 is always negative which satisfies the condition for the existence of localised solitons at the small-amplitude limit of the ion acoustic waves.

In conclusion, we have shown that a nonlinear ion acoustic wave propagates as a localised solitary wave in a warm magnetoplasma with a subsonic speed. The problem of stability of these waves is beyond the scope of this paper and is addressed to future investigation.

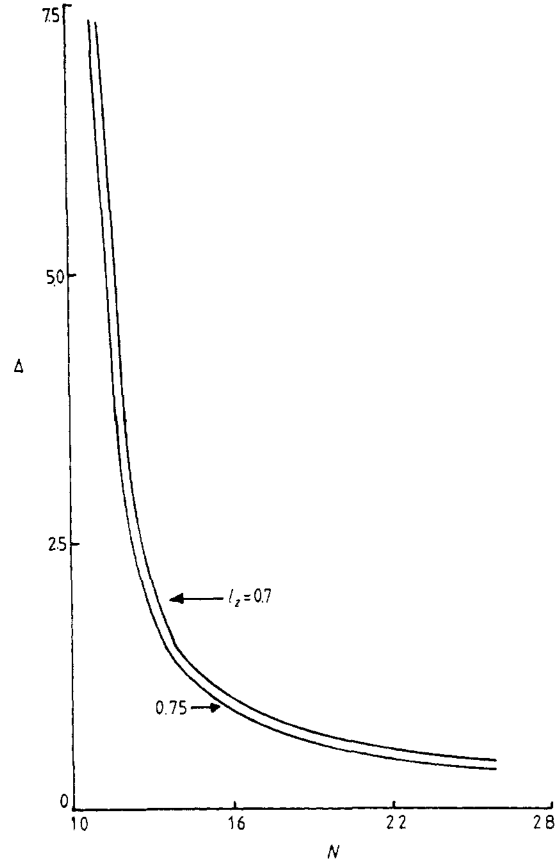

Figure 5. The variation of the width of the soliton with its amplitude at fixed $\sigma = 0.2$ for different values of l_z.

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